EXACT CALCULATION OF THE UNCERTAINTY ON THE INPUT REFLECTION COEFFICIENT OF ARBITRARY TWO-PORTS, DUE TO MISMATCHES AND ARBITRARY REFERENCE PLANES

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ABSTRACT

Exact formulas are derived for the upper and lower limits of the amplitude of the input reflection coefficient of an arbitrary two-port, in the presence of mismatched source and load and/or indefinite reference planes. The derivation is based on certain properties of the bilinear transformation.

Introduction

It was shown by Bandler, Liu and Tromp[1] that, in the realistic worst-case tolerance analysis of microwave two-ports, the effect of a mismatched source and load should not be neglected, as compared with the effect of physical tolerances and model uncertainties. Explicit formulas were derived for the extrema of the modulus of the input reflection coefficient of a lossless two-port, referred to real normalisation impedances. In this paper, we intend to generalise these formulas. We will consider the situation depicted in fig. 1. The S-matrix of the two-port is referred to Z_1' and Z_2'. Source and load impedances Z_S and Z_L are represented by their reflection coefficients, ρ_S and ρ_L, w.r.t. Z_1 and Z_2 respectively. Z_i and Z_i' may be complex. Let

$$\rho_S = |\rho_S| e^{j\phi_S} \quad , \quad \rho_L = |\rho_L| e^{j\phi_L} \qquad (1)$$

We assume ϕ_S, ϕ_L arbitrary, and $|\rho_S|$, $|\rho_L|$ either given, or limited by

$$0 \leq |\rho_S| \leq |\rho_S|^+, \quad 0 \leq |\rho_L| \leq |\rho_L|^+ \qquad (2)$$

This corresponds to the realistic situation, where only a (maximum) VSWR is specified for source and load. We are interested in the input reflection coefficient,

$$\rho_{in} = \frac{Z_{in} - Z_S^*}{Z_{in} + Z_S} \qquad (3)$$

and we shall derive expressions for

$$|\rho_{in}|_M = \max_{\phi_S, \phi_L} |\rho_{in}| \quad , |\rho_{in}|_m = \min_{\phi_S, \phi_L} |\rho_{in}| , \qquad (4)$$

$$\text{or } |\rho_{in}|^+ = \max_{|\rho_S|, |\rho_L|, \phi_S, \phi_L} |\rho_{in}| ,$$

$$|\rho_{in}|^- = \min_{|\rho_S|, |\rho_L|, \phi_S, \phi_L} |\rho_{in}| \qquad (5)$$

Another situation which shall be dealt with is shown in fig. 2, where the lengths of the connecting lines at input and output (given by phase-angles ϕ_1 and ϕ_2) are arbitrary. We shall consider only real Z_i, Z_i' in this case. Again, expressions for the limits (4)-(5) will be derived, but now with the extrema taken over

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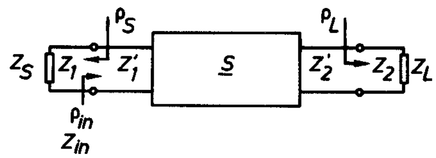

Fig.1. Two-port with mismatched source and load

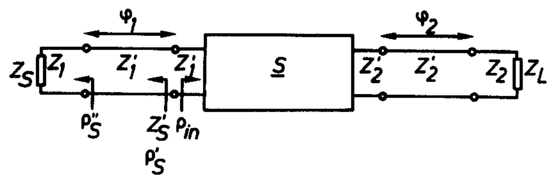

Fig.2. Two-port with arbitrary reference planes

ϕ_1 and ϕ_2 also.

The expressions to be derived are useful for worst-case analysis and fit into the general formulation of the tolerance problem, as given by Tromp[2]. They also give the possibility, when performing a tolerance optimization (see e.g. [1,3-5]) to compromise between the tolerances within the network (i.e. its cost) and the quality of source and load. To a certain extent they can be used to design subnetworks of a large network separately. Finally, they offer an alternative for the study of the stability of a two-port under various conditions of source and load, as well as various positions of input and output reference planes.

Bilinear transformations

Consider the transformation (w,z,a,b,c,d complex)

$$w = \frac{az+b}{cz+d} \qquad (6)$$

This well-known bilinear transformation was, among others, studied by Deschamps[6,7] and is known to transform circles into circles. Without loss of generality, we can consider

$w = Re^{j\phi}$, $0 \leq \phi \leq 2\pi$.

Then

$$z = z_o + r\, e^{j\theta} \;, \; 0 \leq \theta \leq 2\pi \qquad (7)$$

with

$$z_o = \frac{R^2 c^* d - a^* b}{|a|^2 - R^2 |c|^2} \; , \quad r = \frac{R|ad-bc|}{\left||a|^2 - R^2|c|^2\right|} \qquad (8)$$

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Then

$$|z_M| = \operatorname*{Max}_{\phi} \; |z| = |z_o| + r \qquad (9)$$

$$|z_m| = \operatorname*{Min}_{\phi} \; |z| = \big| |z_o| - r \big| \qquad (10)$$

One can prove that $|z_M|$ and $|z_m|$, as functions of R, behave as follows : $|z_M|$ increases to ∞ for $R \leqslant \left|\frac{a}{c}\right|$ and decreases for $R > \left|\frac{a}{c}\right|$, while $|z_m|$ decreases to 0 for $R \leqslant \left|\frac{b}{d}\right|$ and increases for $R > \left|\frac{b}{d}\right|$.

If we let $R^- \leqslant R \leqslant R^+$, then

$$|z|^+ = \operatorname*{Max}_{\phi,R} \; |z| = \begin{cases} |z_M(R^+)| & \text{if } R^+ < \left|\frac{a}{c}\right| \\ \infty & \text{if } R^- \leqslant \left|\frac{a}{c}\right| \leqslant R^+ \\ |z_M(R^-)| & \text{if } R^- > \left|\frac{a}{c}\right| \end{cases} \qquad (11)$$

$$|z|^- = \operatorname*{Min}_{\phi,R} \; |z| = \begin{cases} |z_m(R^+)| & \text{if } R^+ < \left|\frac{b}{d}\right| \\ 0 & \text{if } R^- \leqslant \left|\frac{b}{d}\right| \leqslant R^+ \\ |z_m(R^-)| & \text{if } R^- > \left|\frac{b}{d}\right| \end{cases} \qquad (12)$$

Two special cases of (9) and (10) are of interest

1) If arg a + arg d = arg b + arg c + π, then

$$|z_M| = \left| \frac{|b|+R|d|}{|a|-R|c|} \right| , \quad |z_m| = \left| \frac{|b|-R|d|}{|a|+R|c|} \right| \qquad (13)$$

2) If arg a + arg d = arg b + arg d, then the value of $|z_M|$ and $|z_m|$ is given by Table 1, where

	$\left\|\frac{b}{a}\right\| < \left\|\frac{d}{c}\right\|$		$\left\|\frac{b}{a}\right\| > \left\|\frac{d}{c}\right\|$	
	$R < R_o$	$R > R_o$	$R < R_o$	$R > R_o$
z_M	z_B	z_A	z_A	z_B
z_m	z_A	z_B	z_B	z_A

Table 1

z_M and z_m when arg a + arg d = arg b + arg c

$$|z_A| = \left| \frac{|b|-R|d|}{|a|-R|c|} \right| , \quad |z_B| = \left| \frac{|b|+R|d|}{|a|+R|c|} \right| \qquad (14)$$

and

$$R_o = \sqrt{\left|\frac{ab}{cd}\right|} \qquad (15)$$

We consider also the transformation

$$z = \frac{w^\star d^\star - b^\star}{a - wc} \qquad (16)$$

which does not transform circles into circles, but gives the same $|z|$ as (6). This means that all results, (9) to (15) also apply for (16), which we therefore call the pseudo--bilinear transformation.

Effect of mismatches

In fig. 1, we have

$$|\rho_{in}| = \left| \frac{\rho - \rho_S^\star}{1 - \rho\rho_S} \right| \qquad (17)$$

with

$$\rho = \frac{Z_{in} - Z_1}{Z_{in} + Z_1^\star} \qquad (18)$$

We can now use (16), to find the extrema of $|\rho_{in}|$, for all ρ_S, and either for a given ρ, or for $|\rho|$ between upper and lower limits. ρ depends only on ρ_L, which means that $|\rho|$ can be forced to its extreme values, independently of ρ_S.

Indeed,

$$\rho_L = \frac{a\,\rho + b}{c\,\rho + d} \qquad (19)$$

with

$$a = (1 + S'_{11}\Delta_1)(1 + S'_{22}\Delta_2) - S'_{12}S'_{21}\Delta_1\Delta_2 \qquad (20)$$

$$b = -(\Delta_1^\star + S'_{11})(1 + S'_{22}\Delta_2) + S'_{12}S'_{21}\Delta_2 \qquad (21)$$

$$c = (1 + S'_{11}\Delta_1)(\Delta_2^\star + S'_{22}) - S'_{12}S'_{21}\Delta_1 \qquad (22)$$

$$d = -(\Delta_1^\star + S'_{11})(\Delta_2^\star + S'_{22}) + S'_{12}S'_{21} \qquad (23)$$

$$S'_{11} = \gamma_1^\star S_{11}, S'_{22} = \gamma_2^\star S_{22} \; , \; S'_{12}S'_{21} = \gamma_1^\star\gamma_2^\star S_{12}S_{21} \qquad (24)$$

$$\gamma_i = \frac{Z_i'^\star + Z_i^\star}{Z_i' + Z_i} \; , \quad \Delta_i = \frac{Z_i' - Z_i^\star}{Z_i' + Z_i} \; , \; i = 1,2 \qquad (25)$$

(19) is a bilinear transformation and (9) to (12) give the limits of $|\rho|$, for all ρ_L.

Finally, we find

$$|\rho_{in}|_M = \begin{cases} K^+(|\rho_M|,|\rho_S|) & \text{if } |\rho_M| < 1 \leqslant \frac{1}{|\rho_S|} \\ K^-(|\rho_M|,|\rho_S|) & \text{if } |\rho_M| < \frac{1}{|\rho_S|} \leqslant 1 \\ & \text{or if } 1 \leqslant |\rho_M| < \frac{1}{|\rho_S|} \\ K^+(|\rho_m|,|\rho_S|) & \text{if } \frac{1}{|\rho_S|} \leqslant 1 < |\rho_m| \\ K^-(|\rho_m|,|\rho_S|) & \text{if } \frac{1}{|\rho_S|} < |\rho_m| \leqslant 1 \\ & \text{or if } 1 \leqslant \frac{1}{|\rho_S|} < |\rho_m| \\ \infty & \text{if } |\rho_m| \leqslant \frac{1}{|\rho_S|} \leqslant |\rho_M| \end{cases} \qquad (26)$$

and similar expressions for $|\rho_{in}|_m$, with $|\rho_M|$, $|\rho_m|$ according to (9) and (10).

Also

$$|\rho_{in}|^+ = \begin{cases} K^+(|\rho|^+,|\rho_S|^+) & \text{if } |\rho_L|^+ < \left|\frac{a}{c}\right| \\ & \text{and } |\rho|^+ < 1 \leqslant \frac{1}{|\rho_S|^+} \\ K^-(|\rho|^+,|\rho_S|^+) & \text{if } |\rho_L|^+ < \left|\frac{a}{c}\right| \text{ and} \\ & \text{either } |\rho|^+ < \frac{1}{|\rho_S|^+} \leqslant 1, \\ & \text{or } 1 \leqslant |\rho|^+ < \frac{1}{|\rho_S|^+} \\ \infty & \text{if } |\rho_L|^+ \geqslant \left|\frac{a}{c}\right| \\ & \text{or } |\rho|^+ \geqslant \frac{1}{|\rho_S|^+} \end{cases} \qquad (27)$$

and similar expressions for $|\rho_{in}|^-$, with $|\rho|^+$

and $|\rho|^-$ according to (11) and (12) (with $R^- = 0$, $R^+ = |\rho_L|^+$). We defined

$$K^+(x_1,x_2) = \left|\frac{x_1+x_2}{1+x_1x_2}\right| , \quad K^-(x_1,x_2) = K^+(x_1,-x_2) \tag{28}$$

Undefinite reference planes

Consider fig. 2, with Z_1 and Z_1' real. Let ρ_S' be the reflection coefficient of Z_S' w.r.t Z_1^*, and ρ_S'' this of Z_S w.r.t Z_1'. Then

$$|\rho_S'| = |\rho_S''| \quad \text{and} \quad \rho_S = \frac{\rho_S''+\Delta_1}{1+\rho_S''\Delta_1} \tag{29}$$

We can use (9) to (12) to find the extrema of $|\rho_S'|$ over all ρ_S. The same can be done at the load. A situation similar to that of the preceding paragraph arises and similar formulas are found.

Lossless two-port

If $Z_1' = Z_1$ (real) and if the network is lossless, special case 2) is found and the formulas are considerably simplified. If Z_S and Z_L are passive, the formulas derived in[1] are found.

Example

As an example, we consider the transistor HP 35821E (bias I_C = 15mA, V_{CE} = 15V). Fig. 3 gives the limits of $|\rho_{in}|$. $|\rho_{in}|_M$ and $|\rho_{in}|^+$ coincide, as well as $|\rho_{in}|_m$ and $|\rho_{in}|^-$. The two-port remains stable. Fig. 4 illustrates the effect of undefinite reference planes. The two-port remains stable. In some cases however, the two-port becomes unstable when the reference planes are made arbitrary. The results were confirmed by a Monte Carlo analysis.

Conclusion

Exact, explicit formulas were derived for the limits of the input reflection coefficient of an arbitrary two-port, under various conditions of source and load. They can easily be implemented in a computer program.

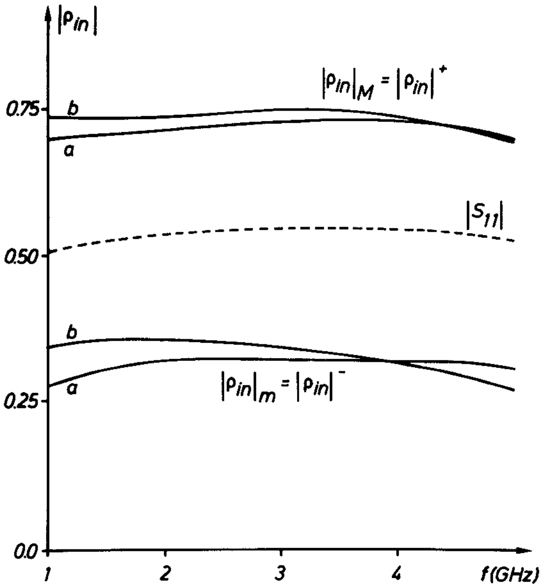

Fig.3. Input reflection coefficient of 35821E, with $|\rho_S|^+ = |\rho_L|^+ = 0.2$ and a) $Z_1 = Z_1' = 50$ b) $Z_1 = 50+0.5j$, $Z_2 = 49-2j$, $Z_1' = 45+5j$, $Z_2' = 55-7j$

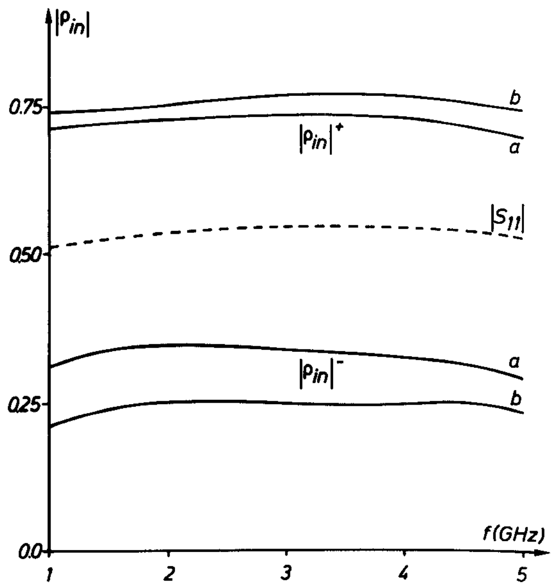

Fig.4. Input reflection coefficient of 35821E, with $|\rho_S|^+ = |\rho_L|^+ = 0.2$, $Z_1 = 50$, $Z_1' = 45$, $Z_2 = 49$, $Z_2' = 55$
a) fixed reference planes
b) indefinite reference planes

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